

Revisiting the Naturalness Problem – Who is afraid of quadratic divergences? –

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Abstract

It is widely believed that quadratic divergences severely restrict *natural constructions* of particle physics models beyond the standard model (SM). Supersymmetry provides a beautiful solution, but the recent LHC experiments have excluded large parameter regions of supersymmetric extensions of the SM. It will now be important to reconsider whether we have been misinterpreting the quadratic divergences in field theories. In this paper, we revisit the problem from the viewpoint of the Wilsonian renormalization group and argue that quadratic divergences, which can always be absorbed into a position of the critical surface, should be simply subtracted in model constructions. Such a picture gives another justification to the argument [5] that the scale invariance of the SM, except for the soft-breaking terms, is an alternative solution to the naturalness problem. It also largely broadens possibilities of model constructions beyond the SM since we just need to take care of logarithmic divergences, which cause mixings of various physical scales and runnings of couplings.

1 Introduction

The hierarchy problem [1], the stability of the weak scale against Planck or GUT scales, is considered to be an important guiding principle to construct a model beyond the standard model (SM). Supersymmetry is a beautiful solution, but the recent LHC experiments have already excluded low energy supersymmetry and we need to solve the little hierarchy problem as well as the μ problem. Faced with these difficulties, it will be important and timely to reconsider the hierarchy problem and ask whether we have been misinterpreting the divergences in field theories.

The hierarchy problem has many faces and it is important to distinguish the following two types. The first one, which is most commonly referred, is why a scalar field can be much lighter than the cutoff scale.¹ The mass of a scalar field receives large radiative corrections due to quadratic divergences, and the tree-level mass of the Higgs field and the loop contributions of the cutoff order must cancel to a very high precision in order of the weak scale. Another type of the hierarchy problem, which is caused by logarithmic divergences, arises when a theory includes multiple physical scales, e.g., the weak scale and the GUT scale [2]. Even if the quadratic divergence is disposed of, the lower mass scale generically receives large radiative corrections of the higher mass scales, and a fine-tuning is necessary to keep the separation of the multiple scales.

The first type of the hierarchy problem can be regarded as an academic question since a subtraction of the quadratic divergences is always possible without any physical effect on low energy dynamics.² It is quite different from the logarithmic divergences, which play important physical roles as beta functions or conformal anomalies. In some formulations, one can sidestep the quadratic divergences and accordingly the associated hierarchy problem. A well-known example is to use the dimensional regularization [3]. Another way may be just to subtract them [4]. A crucial feature in those formulations is that one can separate the subtractive and multiplicative renormalization procedures. Then the first type of the hierarchy problem is reduced to the naturalness of such a subtraction.

As argued in ref. [5], scale invariance can be used to justify a subtracted

¹The meaning of a cutoff is twofold. It is either a cutoff in a UV complete theory or a cutoff in an effective theory at which new degrees of freedom appear. We use it in both meanings and explain their differences depending on the situations.

² In a UV complete theory, quadratic divergence is only an artifact of a regularization procedure while, in an effective theory, it is interpreted as a boundary condition between a UV complete and an effective theory. In both cases, quadratic divergences can be subtracted without any physical effect on low energy dynamics.

theory, analogously to imposing the supersymmetry. At the classical level, the SM is scale invariant except for the Higgs mass term, and a vanishing of the mass term would increase the symmetry of the SM. The common wisdom is that such increase of the symmetry cannot play any role to control the divergences because scale invariance is broken by logarithmic runnings of coupling constants. However, quadratic divergences are independent of the logarithmic divergences and merely an artifact of regularization procedures, and hence should be simply subtracted. Then the trace of the energy-momentum tensor becomes

$$\Theta_\mu^\mu = \Delta m^2 H^\dagger H + \beta_{\lambda_i} \mathcal{O}_i, \quad (1.1)$$

where Δm^2 is *not* proportional to the cutoff squared Λ^2 but to m^2 . The quadratic divergences are subtracted and the mass term gets multiplicatively renormalized. Here, the anomalous term as well as the mass term is regarded as soft-breaking terms of the scale invariance since they do not generate quadratic divergences.

In this paper, we first argue that a subtraction of quadratic divergences is naturally performed from the Wilsonian renormalization group (RG) [6] point of view, and give another justification for a subtracted theory. In the Wilsonian RG, quadratic divergences determine a position of the critical surface in the theory space, and the scaling behavior of RG flows around the critical surface is determined only by the logarithmic divergences. The subtraction of the quadratic divergences can thus be performed by the position of the critical surface. Then, the subtraction is interpreted as a choice of the parameterization, i.e., a coordinate transformation, in the theory space. The fine-tuning we need to perform is the distance of bare parameters from the critical surface, and has nothing to do with the position of the critical surface itself. Hence, quadratic divergences are not the real issue of the fine-tuning problem [7, 8].

If we consider an effective low energy theory with a finite cutoff, we encounter quadratic divergences of the cutoff order. When we embed the effective theory in a UV complete fundamental theory, however, they are compensated by the same kind of divergences arising from the integrations above the cutoff of the effective theory. The subtraction in the effective theory is then justified by the boundary condition at the cutoff. Once the boundary condition is determined by the dynamics in the UV complete fundamental theory, the quadratic divergences that appear in the effective theory can be legitimately subtracted by the boundary condition. One can also use the above arguments of coordinate transformation in the theory space to justify the subtraction within the effective theory.

The second type of the hierarchy problem is more physical and should be

taken with much care. In the Wilsonian RG, it is formulated as a radiative mixing of multiple relevant operators, which is caused by logarithmic divergences. The lower mass scale is affected by higher scales through RG transformations. The mixing is physical and of course cannot be simply subtracted. Hence it gives a strong constraint on *natural* model building.

The above arguments broaden possibilities of model constructions beyond the SM. In particular, nonsupersymmetric models with quadratic divergences but no large logarithmic mixings can be good candidates of models beyond the SM. Examples of such models are ν MSM [9] or a classically conformal TeV scale $B - L$ model [10].

The paper is organized as follows. We first discuss quadratic divergences in the Wilsonian RG in section 2, by studying a ϕ^4 field theory at the one-loop order. We then argue that the quadratic divergence is naturally subtracted and is not the real issue of the fine-tuning problem in section 3. We further discuss logarithmic divergences and the second type of the hierarchy problem in section 4. Finally in section 5, we show that our statements hold at all orders in perturbations. The last section is devoted to conclusions and discussions.

2 RG flows of ϕ^4 theory at one-loop

In this section we explain the role of quadratic divergences in the Wilsonian renormalization group (RG). As an example, we consider a scalar field theory in $d=4$ at the one-loop approximation. The theory has both of the quadratic and logarithmic divergences, but the quadratic divergences can be completely absorbed into the position of the critical surface. In section 5 we see that it holds generally at all orders in perturbation expansions.

We first consider a single scalar theory with a ϕ^4 interaction on a d -dimensional Euclidean lattice. Its action is given by

$$S = \int_{\Lambda^d} \frac{d^d p}{(2\pi)^d} \frac{1}{2} (p^2 + m^2) \phi(p) \phi(-p) + \frac{1}{4!} \lambda \int_{\Lambda^d} \prod_{a=1}^4 \frac{d^d p_a}{(2\pi)^d} (2\pi)^d \delta^{(d)} \left(\sum_{a=1}^4 p_a \right) \phi(p_1) \phi(p_2) \phi(p_3) \phi(p_4) , \quad (2.1)$$

where the momentum integration is performed over the region

$$\Lambda^d = \{p \mid -\pi < p^i < \pi, \forall i = 1, 2, \dots, d\} . \quad (2.2)$$

All the quantities in the action, i.e., the parameters m^2 and λ , the scalar field ϕ , and the cutoff $\Lambda = \pi$, are dimensionless.

An RG transformation can be defined by the following two steps.

Step 1: Integration over higher momentum modes

For simplicity, we introduce a sharp boundary at $p = \pi/N$ with a constant $N > 1$ and divide the integration region Λ^d into two regions with lower and higher momenta,

$$\Lambda_{\text{in}}^d = \{p \mid -\frac{\pi}{N} < p^i < \frac{\pi}{N}, \forall i = 1, 2, \dots, d\}, \quad (2.3)$$

$$\Lambda_{\text{out}}^d = \{p \mid |p^i| \geq \frac{\pi}{N}, \exists i = 1, 2, \dots, d\}. \quad (2.4)$$

We then perform functional integrations over $\phi(p)$ with $p \in \Lambda_{\text{out}}^d$. The remaining theory is described by an effective action for the lower momentum modes, $\phi(p)$ with $p \in \Lambda_{\text{in}}^d$.

Step 2: Rescalings

We then rescale the momentum p and the field $\phi(p)$ as

$$\begin{aligned} p' &= Np, \\ \phi'(p') &= N^{-\theta} \phi(p). \end{aligned} \quad (2.5)$$

This rescaling of momenta makes the integration region back to the original one, Λ^d . The scaling dimension θ can be chosen so that the scalar field has the canonical kinetic term. $(-\theta)$ is the mass dimension of the field $\phi(p)$ and given by the canonical value $\theta = (d+2)/2$ near the Gaussian fixed point.³

In the $d=4$ ϕ^4 theory, we can restrict RG transformations within the space of two parameters m^2 and λ . The other operators, such as ϕ^6 or $p^4\phi^2$, become less and less important when repeating the RG transformations and eventually become negligible: they are irrelevant operators. Hence, the restriction to consider the RG transformations within the subspace is justified. The resulting theory has the same form as the original one in (2.1), but with the parameters changed.

We perform the above functional integrations over $\phi(p)$ with $p \in \Lambda_{\text{out}}^d$ by perturbative expansions with respect to the coupling λ . At the one-loop order, the above two steps give the following changes of the parameters:

$$m^{2'} = N^{2\theta-d}(m^2 + c_1\lambda - c_2m^2\lambda), \quad (2.6)$$

$$\lambda' = N^{4\theta-3d}(\lambda - 3c_2\lambda^2), \quad (2.7)$$

³ If we consider RG flows near a nontrivial fixed point, θ deviates from the canonical value due to an anomalous dimension of the field. Near the Gaussian fixed point, we can keep θ to be the canonical one and absorb all effects of the wave function renormalization in the coefficients c_n given below. At one-loop of the ϕ^4 theory, a wave function renormalization is absent.

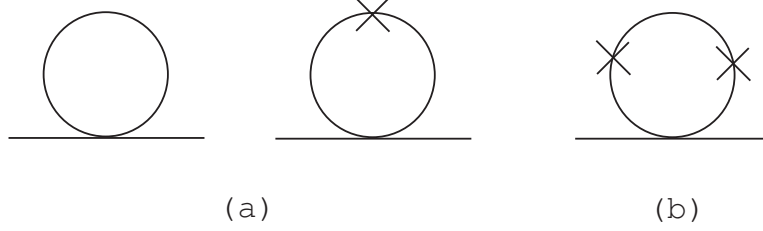


Figure 1: (a) Feynman diagrams that contribute to the mass renormalization transformation (2.6). The cross represents a mass insertion. (b) A diagram in higher order in m^2 , which does not contribute to (2.6) in the limit (2.10).

where c_1 and c_2 are positive constants,

$$c_1 = \frac{1}{2} \int_{\Lambda_{\text{out}}^d} \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} > 0, \quad (2.8)$$

$$c_2 = \frac{1}{2} \int_{\Lambda_{\text{out}}^d} \frac{d^d q}{(2\pi)^d} \left(\frac{1}{q^2} \right)^2 > 0. \quad (2.9)$$

The prefactors $N^{2\theta-d}$ and $N^{4\theta-3d}$ come from the rescalings (Step 2), while c_n are contributions from the integrations of higher momentum modes (Step 1). The mass transformation (2.6) is given by the diagrams in Figure 1(a). Since the integration is performed in the UV region Λ_{out}^d and no IR divergences occur, we can expand the propagator with respect to m^2 . In eq. (2.6) we took the first two terms in the expansion. Higher order terms in m^2 , such as $c'_2 m^4 \lambda$ which comes from a diagram in Figure 1(b), are highly suppressed since we suppose that

$$\frac{m^2}{\Lambda^2} \ll 1. \quad (2.10)$$

Similarly the coupling transformation (2.7) at one-loop is determined by the diagram in Figure 2(a). Again, higher order terms in m^2 , such as $c'_2 m^2 \lambda^2$ for Figure 2(b), do not contribute to (2.7) due to the assumption (2.10). There are no wave function renormalizations at the one-loop order. The dependence of the constants c_n on the cutoff Λ is easily derived.⁴ By evaluating the integrals of (2.8) and (2.9), we find

$$c_1 \propto \Lambda^{d-2}(1 - N^{-(d-2)}) \longrightarrow \Lambda^2(1 - N^{-2}) \text{ for } d = 4, \quad (2.11)$$

$$c_2 \propto \Lambda^{d-4}(1 - N^{-(d-4)}) \longrightarrow \Lambda^0 \ln N \text{ for } d = 4. \quad (2.12)$$

⁴In taking the continuum limit, we consider a limit where the physical momentum becomes infinitesimal compared to the cutoff $\Lambda = \pi$. Hence, $\Lambda = \pi$ corresponds to a large momentum scale in the unit of the physical scale.

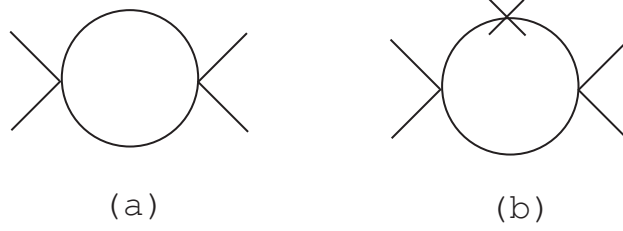


Figure 2: (a) A Feynman diagram that contributes to the coupling renormalization transformation (2.7). (b) A diagram in higher order in m^2 , which does not contribute to (2.7) in the limit (2.10).

Hence, c_1 and c_2 reflect the quadratic and the logarithmic divergences, respectively.

By performing the RG transformations (2.6) and (2.7) several times, one obtains RG flows in the theory space, the space spanned by the parameters λ and m^2 in this case. Since eq. (2.7) depends only on λ , but not on m^2 , the flow of the coupling constant λ is determined only by using (2.7). For $d \neq 4$, the equation (2.7) is rewritten as

$$\frac{1}{\lambda'} - \frac{1}{\lambda^*} = N^{-(4\theta-3d)} \left(\frac{1}{\lambda} - \frac{1}{\lambda^*} \right) + \mathcal{O}(\lambda), \quad \lambda^* = \frac{N^{4\theta-3d} - 1}{3c_2}. \quad (2.13)$$

After performing the transformation n times, one obtains

$$\frac{1}{\lambda_n} - \frac{1}{\lambda^*} = N^{-(4\theta-3d)n} \left(\frac{1}{\lambda_0} - \frac{1}{\lambda^*} \right). \quad (2.14)$$

Here, λ_n is the renormalized coupling constant after n -times RG transformations. There are two fixed points of the RG transformation: one with $\lambda = 0$, and the other with $\lambda = \lambda^*$. For $d = 4$, two fixed points coincide at $\lambda = 0$ and one finds

$$\frac{1}{\lambda_n} = \frac{1}{\lambda_0} + 3c_2 n. \quad (2.15)$$

If $\lambda_0 > 0$, λ_n approaches the fixed point $\lambda = 0$ as one increases n .

In order to obtain the flow in the direction of m^2 , we rewrite eq. (2.6), by using (2.7), as

$$m'^2 - m_c^2(\lambda') = N^{2\theta-d} (1 - c_2 \lambda) (m^2 - m_c^2(\lambda)), \quad (2.16)$$

with a function $m_c^2(\lambda)$, determined up to this order as

$$m_c^2(\lambda) = -\frac{c_1}{1 - N^{2(\theta-d)}} \lambda. \quad (2.17)$$

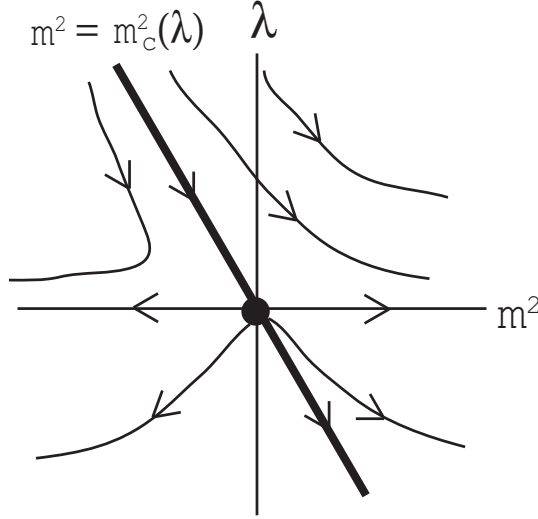


Figure 3: RG flow for $d = 4$. The blob indicates the Gaussian fixed point, and the thick line $m^2 = m_c^2(\lambda)$ corresponds to the critical line.

Performing the RG transformation n times, one obtains

$$\begin{aligned}
 m_n^2 - m_c^2(\lambda_n) &= N^{(2\theta-d)n} \prod_{i=0}^{n-1} (1 - c_2 \lambda_i) (m_0^2 - m_c^2(\lambda_0)) \\
 &\simeq N^{(2\theta-d)n} \exp(-c_2 \sum_{i=0}^{n-1} \lambda_i) (m_0^2 - m_c^2(\lambda_0)) . \quad (2.18)
 \end{aligned}$$

The equation $m^2 = m_c^2(\lambda)$ determines the position of the critical line, and eq. (2.18) shows how the *distance* from the critical line scales under the RG transformations.

The RG flow is given by the two equations (2.18) and (2.14), or (2.15) for $d = 4$. A schematic picture of the flow for the $d = 4$ case is drawn in Figure 3. The fixed point of the RG flow is given by $m^2 = \lambda = 0$, i.e., the Gaussian fixed point. If the initial parameters are exactly on the critical line $m^2 = m_c^2(\lambda)$, they continue to be there after RG transformations and approach the fixed point. We consider only the region $\lambda \geq 0$ for the stability of vacuum. A theory on the critical line is a massless theory. If the initial parameters are off the critical line, they depart from it with λ decreasing under the RG transformations.

As we stressed in the introduction, the constant c_1 , which reflects the quadratic divergences, is completely absorbed into the position of the critical line, i.e., the definition of the function $m_c^2(\lambda)$ in (2.17), and the scaling behavior (2.18) of the RG flow around the critical line is determined only by c_2 , which reflects the logarithmic divergences. It is a very important message we

can read from the Wilsonian treatment of the renormalization group. As we see in the next section, it corresponds to the fact that in ordinary perturbative calculations, quadratic divergences can always be subtracted without any physical effects on the dynamics of field theories, unlike logarithmic divergences.

3 Quadratic divergences, subtractions, and the fine-tuning problem

We now interpret the subtraction and the fine-tuning problem in the framework of the Wilsonian RG, and argue that the quadratic divergence can be naturally subtracted and is not the real issue of the hierarchy problem.

3.1 Continuum limit

We first review how to take the continuum limit. In the constructive formulation of a field theory, one usually takes the continuum limit by simultaneously letting the parameters close to the critical surface and taking the cutoff to infinity. For a scalar theory in $d = 4$, however, one cannot construct an interacting theory in this way, which is well-known as the triviality or the Landau singularity problem. We thus consider a theory at a large but finite cutoff with bare parameters very close to the critical surface.

We first write the RG equation (2.18) as

$$m_0^2 - m_c^2(\lambda_0) = N^{-(2\theta-d)n} e^{c_2 \sum_{i=0}^{n-1} \lambda_i} (m_n^2 - m_c^2(\lambda_n)) . \quad (3.1)$$

The parameters m_0^2 and λ_0 on the left-hand side (LHS) describe the bare parameters, while m_n^2 and λ_n on the right-hand side (RHS) describe the renormalized ones. Note that all the quantities in this relation are dimensionless, including the cutoff $\Lambda = \pi$. We now introduce dimensionful cutoffs. First we introduce the low energy scale $\tilde{\Lambda}_n = M$, e.g., $M = 100$ GeV. The renormalized parameters m_n and λ_n are defined at this scale. The higher scale where the bare parameters m_0, λ_0 are defined is given by

$$\tilde{\Lambda}_0 = N^n \tilde{\Lambda}_n = N^n M . \quad (3.2)$$

Let us express everything in terms of dimensionful physical quantities instead of the dimensionless lattice parameters. Physical momentum \tilde{p} is defined by

$$\tilde{p}_k = \frac{\tilde{\Lambda}_k}{\Lambda} p , \quad \tilde{\Lambda}_k = N^{n-k} M , \quad (3.3)$$

for each cutoff theory at $k = 0, 1, \dots, n$. We can similarly define the field as $\tilde{\phi}_k(\tilde{p}_k) = (\tilde{\Lambda}_k/\Lambda)^{-\frac{d+2}{2}}\phi(p)$. The dimensionless parameters m^2, λ are also replaced by a dimensionful mass and a coupling

$$\tilde{m}_k^2 = \left(\frac{\tilde{\Lambda}_k}{\Lambda}\right)^2 m^2, \quad \tilde{\lambda}_k = \left(\frac{\tilde{\Lambda}_k}{\Lambda}\right)^{4-d} \lambda. \quad (3.4)$$

In terms of these physical quantities, the RG equation (3.1) can be rewritten as

$$\tilde{m}_0^2 - \tilde{m}_c^2(\lambda_0) = N^{(2-2\theta+d)n} e^{c_2 \sum_{i=0}^{n-1} \lambda_i} (\tilde{m}_n^2 - \tilde{m}_c^2(\lambda_n)). \quad (3.5)$$

The critical value $m_c^2(\lambda)$ is also rescaled as

$$\tilde{m}_c^2(\lambda_k) = \left(\frac{\tilde{\Lambda}_k}{\Lambda}\right)^2 m_c^2(\lambda_k), \quad (3.6)$$

which reflects quadratic divergences of the position of the critical line, as shown in the previous section. For the Gaussian fixed point, we have $\theta = (d+2)/2$, and hence $N^{(2-2\theta+d)n} = 1$. Eq. (3.5) becomes

$$(\tilde{m}_0^2 - \tilde{m}_c^2(\lambda_0)) = e^{c_2 \sum_{i=0}^{n-1} \lambda_i} (\tilde{m}_n^2 - \tilde{m}_c^2(\lambda_n)). \quad (3.7)$$

The prefactor $e^{c_2 \sum_{i=0}^{n-1} \lambda_i}$ represents the logarithmic running of the mass parameter. Indeed, from (2.15), one can find $\lambda_i \sim 1/3c_2 i$, and thus the prefactor behaves as $e^{c_2 \sum_{i=0}^{n-1} \lambda_i} \sim n^{1/3} \sim (\ln \tilde{\Lambda}_0)^{1/3}$. Its n -dependence is weaker than a power-law running behavior $N^{xn} \sim (\tilde{\Lambda}_0)^x$ with some constant x .

In ordinary perturbative calculations of field theories, we first choose a regularization method and a renormalization prescription to deal with divergences. Quadratic divergences are simply subtracted with appropriate renormalization conditions. The so called fine-tuning problem is that we need to fine-tune the bare mass \tilde{m}_0^2 of a scalar field at the cutoff scale $\tilde{\Lambda}_0$, so that the renormalized mass \tilde{m}_R^2 ,

$$\begin{aligned} \tilde{m}_R^2 &= \tilde{m}_0^2 + \alpha(\lambda, \ln(\tilde{\Lambda}_0/\tilde{\Lambda}_n)) \tilde{\Lambda}_0^2 + \beta(\lambda, \ln(\tilde{\Lambda}_0/\tilde{\Lambda}_n)) \tilde{m}_0^2 \\ &= (1 + \beta(\lambda, \ln(\tilde{\Lambda}_0/\tilde{\Lambda}_n))) (\tilde{m}_0^2 + \alpha'(\lambda, \ln(\tilde{\Lambda}_0/\tilde{\Lambda}_n)) \tilde{\Lambda}_0^2), \end{aligned} \quad (3.8)$$

is held at the desired value. Here α and β are functions of the coupling λ and $\ln(\tilde{\Lambda}_0/\tilde{\Lambda}_n)$, and assumed to be independent of the mass parameter \tilde{m}_0 . Namely, the mass-independent renormalization scheme [11, 12] is chosen. When we apply (3.8) to the Higgs mass, this looks very unnatural because, if $\tilde{\Lambda}_0 \gg \tilde{m}_R = m_W$, the tree-level mass \tilde{m}_0^2 of the Higgs particle and the loop contributions $\alpha \tilde{\Lambda}_0^2$ must cancel to a very high precision in order of the weak scale m_W^2 . There

is also a subtlety in the mass-independent renormalization for theories with quadratic divergences [12, 4].

Let us make a comparison between eq. (3.7) and eq. (3.8). The bare mass parameter \tilde{m}_0^2 and the quadratic divergence $\alpha'\tilde{\Lambda}_0^2$ in the RHS of (3.8) correspond to the bare mass \tilde{m}_0^2 and the critical value $\tilde{m}_c^2(\lambda_0)$ in the LHS of (3.7), respectively. The renormalized mass \tilde{m}_R^2 of (3.8) is given by $\tilde{m}_n^2 - \tilde{m}_c^2(\lambda_n)$ in the RHS of (3.7). The multiplicative renormalization factor $(1 + \beta)$ in (3.8) corresponds to the factor for the logarithmic running $e^{-c_2 \sum_{i=0}^{n-1} \lambda_i}$ in (3.7).

3.2 Subtractive renormalization in Wilsonian RG

In the Wilsonian RG, the quadratic divergence c_1 is absorbed into the position of the critical line $\tilde{m}_c^2(\lambda)$. Note also that all observable quantities like the correlation length are determined by the distance of the mass parameter \tilde{m}_0^2 from the critical line $\tilde{m}_c^2(\lambda_0)$, not by the value \tilde{m}_0^2 itself. The fact that the difference $\tilde{m}_0^2 - \tilde{m}_c^2(\lambda_0)$ is the physically relevant quantity gives a natural interpretation for the subtractive renormalization of the quadratic divergences.

As we stressed in the introduction, an important and necessary feature for the subtracted theories, as in [3, 4, 5], where one can sidestep the quadratic divergences, is that one can separate the subtractive and the multiplicative renormalization procedures. Looking at eq. (3.7), the prefactor $e^{c_2 \sum_{i=0}^{n-1} \lambda_i}$ corresponds to the multiplicative renormalization. Hence, eq. (3.7) clearly separates the subtractive, $\tilde{m}_0^2 - \tilde{m}_c^2(\lambda_0)$, and the multiplicative, $e^{c_2 \sum_{i=0}^{n-1} \lambda_i}$, renormalization procedures.

Another remarkable result from eq. (3.7) is as follows. Neither the prefactor $e^{c_2 \sum_{i=0}^{n-1} \lambda_i}$ nor the critical value $\tilde{m}_c^2(\lambda)$ depends on the mass parameter \tilde{m}_0 . Hence, eq. (3.7) gives an explicit realization of the mass-independent renormalization scheme [11, 12]. As discussed in [12, 4], whether or not one can formulate a mass-independent renormalization scheme in theories with quadratic divergences was a nontrivial and subtle problem.

Note that the subtraction is not arbitrary in the Wilsonian RG. Once we choose a scheme to calculate the RG flow, the critical line is given unambiguously. Seeming ambiguities only come from our lack of calculability of the exact RG flows, but RG flows are in principle exactly determined once we choose a scheme. The only fine-tuning we need is the *distance* from the critical surface to the bare parameters.

Before discussing the tuning of the distance, let us convince ourselves that the position of the critical surface has nothing to do with the fine-tuning problem by comparing two theories: one with a nonzero c_1 , and the other with a

vanishing c_1 . The latter theory corresponds to a theory with vanishing quadratic divergences. The critical lines are given by $m_c^2 \neq 0$ for $c_1 \neq 0$, and $m_c^2 = 0$ for $c_1 = 0$. Is the first theory more unnatural than the second one? The feeling of unnaturalness simply comes from seeming ambiguities of calculating the position of the critical line, but as stressed, once a scheme is given it is determined unambiguously and there are no distinctions between these two theories. In other words, we can take new coordinates of the theory space such that the critical line is parallel to an axis of the coordinates. Then a theory with $c_1 \neq 0$ looks the same as a theory with $c_1 = 0$. The change of coordinates can be determined exactly once we choose a scheme. In the dimensional regularization, quadratic divergences do not appear. We may say that this corresponds to taking a new coordinate by $m_{\text{new}}^2 = m^2 - m_c^2(\lambda)$. Then the critical line is given by $m_{\text{new},c}^2(\lambda) = 0$ even for a theory with $c_1 \neq 0$. Therefore the subtraction of the quadratic divergences is simply a choice of the coordinates of the theory space. In this sense, the quadratic divergences are naturally subtracted and are not the real issue of the hierarchy problem.

Finally, we consider the tuning of the distance from the critical line, namely, the fine-tuning to choose the bare mass near the critical value with the precision of the weak scale. Such a fine-tuning is kinematical in the following sense. Looking at eq. (3.1), we see that this kind of tuning comes from the factor $N^{-(2\theta-d)n} = N^{-2n}$ for the Gaussian fixed point, but it reflects nothing but the canonical dimension of the mass and has nothing to do with the quadratic divergences. This fine-tuning is necessary as long as the dimension of mass square is two. The prefactor $e^{c_2 \sum \lambda_i}$ simply gives a logarithmic scaling factor $n^{1/3}$, and does not change the situation much.

If we consider a nontrivial fixed point instead of the Gaussian one, the canonical scaling dimension is modified by anomalous dimensions of the mass and the field. For example, if $\theta = (d+2)/2 - \delta$ and the coupling constant at the fixed point is given by λ_* , the relation (3.7) is modified as⁵

$$(\tilde{m}_0^2 - \tilde{m}_c^2(\lambda_0)) \sim (N^{2\delta} e^{c_2 \lambda_*})^n (\tilde{m}_n^2 - \tilde{m}_c^2(\lambda_n)) . \quad (3.9)$$

Thus, large anomalous dimensions can relax the condition of how precisely we need to define the bare mass near the critical line. If one feels uncomfortable with the relation (3.7), a possible resolution will be to construct a theory with large anomalous dimensions.

⁵ The prefactor $e^{c_2 \sum_{i=0}^{n-1} \lambda_i}$ in (3.7) gives not only the factor $e^{c_2 \lambda_* n}$ in (3.9), but also an extra factor $e^{c_2 \sum_{i=0}^{n-1} (\lambda_i - \lambda_*)}$. Since it has a weaker n -dependence than a power-law behavior $N^{xn} \sim \tilde{\Lambda}^x$, we neglected it here. The wave function renormalization also gives a deviation from θ , δ and an extra factor neglected in (3.9).

4 Mixing of multiple scales

In the previous two sections, we considered a theory with a single physical scale besides the cutoff scale. In this section, we study an RG flow in a theory with hierarchically separated multiple scales. These scales are mixed by radiative corrections associated with the logarithmic divergences. This causes the second type of the hierarchy problem. We emphasize that it again has nothing to do with the quadratic divergences.

For simplicity, we consider a theory with multiple scalar fields on a d -dimensional Euclidean lattice, whose action is given by

$$S = \int_{\Lambda^d} \left[\sum_{\alpha=1}^S \left(\frac{1}{2}(p^2 + m_\alpha^2)\phi_\alpha^2 + \frac{1}{4!}\lambda_{\alpha\alpha}\phi_\alpha^4 \right) + \sum_{\alpha \neq \beta} \frac{1}{8}\lambda_{\alpha\beta}\phi_\alpha^2\phi_\beta^2 \right], \quad (4.1)$$

with $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}$. The index $\alpha = 1, \dots, S$ labels species of the scalar fields. There are S mass parameters m_α^2 and $S(S+1)/2$ coupling parameters $\lambda_{\alpha\beta}$ in this case. The momentum integration region Λ^d is taken as in eq. (2.2).

To obtain an RG transformation, we follow the two steps in section 2. At the one-loop order, it becomes

$$m_\alpha'^2 = N^{2\theta-d} \left[m_\alpha^2 + c_1 \sum_\beta \lambda_{\alpha\beta} - c_2 \sum_\beta \lambda_{\alpha\beta} m_\beta^2 \right], \quad (4.2)$$

$$\lambda'_{\alpha\alpha} = N^{4\theta-3d} \left[\lambda_{\alpha\alpha} - 3c_2 \sum_\beta (\lambda_{\alpha\beta})^2 \right], \quad (4.3)$$

$$\lambda'_{\alpha\beta} = N^{4\theta-3d} \left[\lambda_{\alpha\beta} - c_2 \sum_\gamma \lambda_{\alpha\gamma} \lambda_{\beta\gamma} - 4c_2 (\lambda_{\alpha\beta})^2 \right], \quad (4.4)$$

with c_1 and c_2 given in (2.8) and (2.9).

By performing the RG transformations several times, one obtains RG flows in the theory space, $S(S+3)/2$ -dimensional space spanned by the parameters $\lambda_{\alpha\beta}$ and m_α^2 . We use λ to represent the set of couplings $\lambda_{\alpha\beta}$ in the following. Since eqs. (4.3) and (4.4) depend only on $\lambda_{\alpha\beta}$, and not on m_α^2 , the flow in $\lambda_{\alpha\beta}$ is determined by (4.3) and (4.4). For the flow in m_α^2 , we can rewrite eq. (4.2) as

$$m_\alpha'^2 - m_{c\alpha}^2(\lambda') = N^{2\theta-d} \sum_\beta (\delta_{\alpha\beta} - c_2 \lambda_{\alpha\beta}) (m_\beta^2 - m_{c\beta}^2(\lambda)), \quad (4.5)$$

where $m_{c\alpha}^2(\lambda)$ are functions of $\lambda_{\alpha\beta}$ and are determined by the equations

$$m_{c\alpha}^2(\lambda') - N^{2\theta-d} \sum_\beta (\delta_{\alpha\beta} - c_2 \lambda_{\alpha\beta}) m_{c\beta}^2(\lambda) = N^{2\theta-d} c_1 \sum_\beta \lambda_{\alpha\beta}, \quad (4.6)$$

together with (4.3) and (4.4). The solutions are given as a power series of λ as

$$m_{c\alpha}^2(\lambda) = -\frac{c_1}{1 - N^{2(\theta-d)}} \sum_{\beta} \lambda_{\alpha\beta} + \mathcal{O}(\lambda^2) . \quad (4.7)$$

The quadratic divergence in $d=4$, namely c_1 , is again absorbed into the position of the critical surface $m_{\alpha}^2 = m_{c\alpha}^2(\lambda)$.

Now let us define a symmetric mixing matrix

$$(M_{(k)})_{\alpha\beta} = \delta_{\alpha\beta} - c_2 \lambda_{\alpha\beta(k)} \simeq \exp(\delta_{\alpha\beta} - c_2 \lambda_{\alpha\beta(k)}) , \quad (4.8)$$

and their product

$$M = M_{(n-1)} M_{(n-2)} \cdots M_{(0)} . \quad (4.9)$$

Here $\lambda_{\alpha\beta(k)}$ represents the coupling constant $\lambda_{\alpha\beta}$ after k -times RG transformation. By performing the transformation (4.5) n times, one obtains

$$m_{\alpha(n)}^2 - m_{c\alpha}^2(\lambda_{(n)}) = N^{(2\theta-d)n} \sum_{\beta} M_{\alpha\beta} (m_{\beta(0)}^2 - m_{c\beta}^2(\lambda_{(0)})) . \quad (4.10)$$

In taking the cutoff very large, one needs to fine-tune the S relevant operators, $m_{\alpha(0)}^2$. We rewrite the RG equation (4.10) as

$$m_{\alpha(0)}^2 - m_{c\alpha}^2(\lambda_{(0)}) = N^{-(2\theta-d)n} \sum_{\beta} (M^{-1})_{\alpha\beta} (m_{\beta(n)}^2 - m_{c\beta}^2(\lambda_{(n)})) , \quad (4.11)$$

where

$$(M^{-1})_{\alpha\beta} = \delta_{\alpha\beta} + c_2 \sum_{k=0}^{n-1} \lambda_{\alpha\beta(k)} + \mathcal{O}(\lambda^2) . \quad (4.12)$$

The equation (4.11) can be written in terms of the dimensionful parameters, as in (3.7), as

$$\tilde{m}_{\alpha(0)}^2 - \tilde{m}_{c\alpha}^2(\lambda_{(0)}) = \sum_{\beta} (M^{-1})_{\alpha\beta} (\tilde{m}_{\beta(n)}^2 - \tilde{m}_{c\beta}^2(\lambda_{(n)})) . \quad (4.13)$$

Here we used the canonical value of $\theta = (d+2)/2$ for the Gaussian fixed point.

We now simplify the discussion by considering a theory with two separate renormalized scales, e.g.,

$$\begin{aligned} \tilde{m}_{1(n)}^2 - \tilde{m}_{c1}^2(\lambda_{(n)}) &= m_W^2 , \\ \tilde{m}_{2(n)}^2 - \tilde{m}_{c2}^2(\lambda_{(n)}) &= m_{\text{GUT}}^2 . \end{aligned} \quad (4.14)$$

It then follows from (4.13) that we need to fine-tune the bare mass parameters such that the difference of $(\tilde{m}_{1,0}^2 - \tilde{m}_{c1}^2(\lambda_0))$ and radiative corrections from the higher physical scale m_{GUT}^2 are canceled to give the weak scale:

$$m_W^2 \simeq \frac{1}{(M^{-1})_{11}} (\tilde{m}_{1,0}^2 - \tilde{m}_{c1}^2(\lambda_0)) - \frac{(M^{-1})_{12}}{(M^{-1})_{11}} m_{\text{GUT}}^2 . \quad (4.15)$$

Unlike the subtraction of the critical mass parameter $m_c^2(\lambda)$, the GUT scale m_{GUT}^2 is a physically observable scale and cannot be subtracted. Or, in other words, the second term proportional to m_{GUT}^2 cannot be absorbed into the position of the critical surface. Hence, unless the off-diagonal element of the matrix $(M^{-1})_{12}$ is suppressed, we need a fine-tuning of the bare mass $(\tilde{m}_{1,0}^2 - \tilde{m}_{c1}^2(\lambda_0))$ against the GUT scale with a high precision in order of the weak scale. As we can see from (4.12), in order to solve the problem, we need to suppress either the coefficient c_2 , the mutual couplings $\lambda_{\alpha\beta}$, or the higher scale m_{GUT}^2 .

5 Higher orders of perturbations

Up to now, our discussions are based on the one-loop order calculations, but our statements hold at all orders of perturbations in the coupling λ . In this section, we extend the statement in section 2 to all orders. It is also straightforward to extend the results in the other sections in the same way.

In the following, by using a renormalized perturbation theory, we will show iteratively that the statement in section 2 does hold at all orders of perturbations in the coupling constant λ . We first replace the mass parameter m^2 by $m^2 - m_c^2 + m_c^2$ in the action (2.1),

$$S = \int_{\Lambda} \left[\frac{1}{2} p^2 \phi^2 + \frac{1}{2} (m^2 - m_c^2(\lambda)) \phi^2 + \frac{1}{2} m_c^2(\lambda) \phi^2 + \frac{1}{4!} \lambda \phi^4 \right], \quad (5.1)$$

where the λ dependence of the position of the critical surface $m_c^2(\lambda)$ will be determined later in a self-consistent way. We then perform the functional integrations over the higher momentum modes (Step 1 of the RG transformation) by perturbative expansions with respect to $m^2 - m_c^2(\lambda)$ and $m_c^2(\lambda)$, as well as λ . Such mass expansions are legitimate since the integrations are performed only in the UV region and free from IR divergences. An insertion of the renormalized mass parameter $m^2 - m_c^2(\lambda)$ suppresses loop integrations by a factor $1/\Lambda^2$, and thus always appears in a combination of

$$\frac{m^2 - m_c^2(\lambda)}{\Lambda^2} = \mathcal{O}\left(\frac{1}{\Lambda^2}\right). \quad (5.2)$$

Here, we assumed that $m^2 - m_c^2(\lambda)$ is of order Λ^0 . It was explicitly shown at the one-loop order in sections 2, and will be justified later in this section in an iterative way at higher orders in λ . We can thus neglect higher order terms in the expansion of $m^2 - m_c^2(\lambda)$. On the other hand, an insertion of the critical mass is not suppressed by $1/\Lambda^2$ since the value $m_c^2(\lambda)$ itself is of order Λ^2 :

$$\frac{m_c^2(\lambda)}{\Lambda^2} = \mathcal{O}(\lambda). \quad (5.3)$$

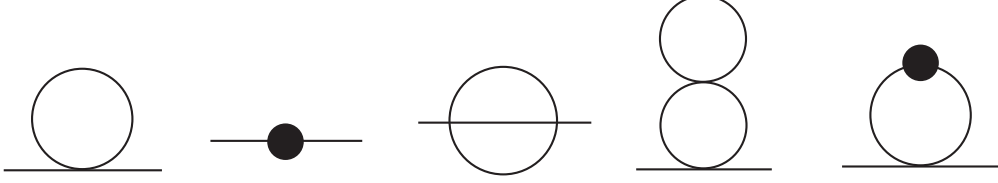


Figure 4: The Feynman diagrams that contribute to $f\Lambda^2$ up to order λ^2 . The dot represents an insertion of m_c^2 . The first two diagrams give order λ^1 contributions, while the last four give order λ^2 . Note that the second diagram gives both order λ^1 and λ^2 contributions.

Though it is suppressed by the coupling constant λ , it is not suppressed by the cutoff Λ . It was already shown in (2.17) at one-loop order, and can be justified later in this section in an iterative way. Hence, on the contrary to (5.2), we need to take higher orders of $m_c^2(\lambda)$ in the perturbative calculations.

As a result of the above arguments, we can schematically write the effective action after the integrations as

$$\int_{\Lambda/N} \left[\frac{1}{2} e\left(\lambda, \frac{m_c^2}{\Lambda^2}\right) p^2 \phi^2 + \frac{1}{2} \left[f\left(\lambda, \frac{m_c^2}{\Lambda^2}\right) \Lambda^2 + g\left(\lambda, \frac{m_c^2}{\Lambda^2}\right) (m^2 - m_c^2(\lambda)) \right] \phi^2 + \frac{1}{4!} h\left(\lambda, \frac{m_c^2}{\Lambda^2}\right) \phi^4 \right], \quad (5.4)$$

where the functions e , f , g and h are power series with respect to λ and m_c^2/Λ^2 , with the coefficients dependent on N . $f\Lambda^2$ is given by the 2-point, 1PI diagrams with the external momentum fixed to be zero. They are depicted in Figure 4 up to order λ^2 , where the dot represents an insertion of m_c^2 . g is given by those diagrams with a single insertion of $m^2 - m_c^2(\lambda)$. They are depicted in Figure 5 up to order λ^2 , where the cross represents an insertion of $m^2 - m_c^2$. $e - 1$ is given by the diagrams of f , but the first term in the expansion of the external momentum. They begin with the third diagram in Figure 4 at order λ^2 . Similarly, h is given by the 4-point, 1PI diagrams with the external momentum fixed to be zero. They are depicted in Figure 6 up to order λ^3 ,

We next perform the wave function renormalization, $\phi \rightarrow e^{-1/2}\phi$, and the rescaling of momenta and fields. It is Step 2 of the RG transformation, and we have

$$\int_{\Lambda} \left[\frac{1}{2} p^2 \phi^2 + \frac{1}{2} N^{2\theta-d} e^{-1} [f \Lambda^2 + g (m^2 - m_c^2(\lambda))] \phi^2 + \frac{1}{4!} N^{4\theta-3d} e^{-2} h \phi^4 \right].$$

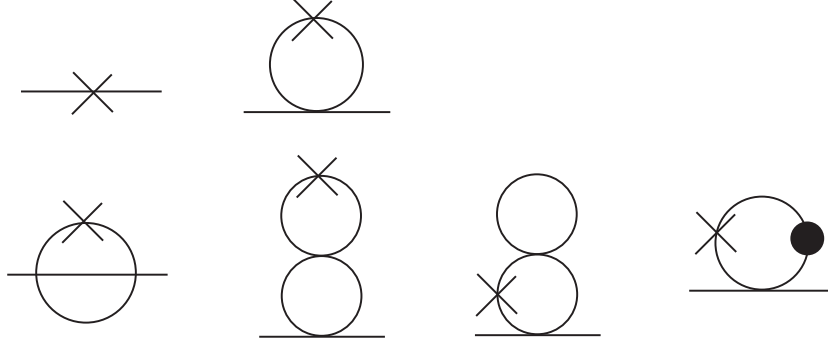


Figure 5: The Feynman diagrams that contribute to g up to order λ^2 . The cross and the dot represent an insertion of $m^2 - m_c^2$ and m_c^2 , respectively. The first diagram gives an order λ^0 contribution, the second λ^1 , and the last four λ^2 .

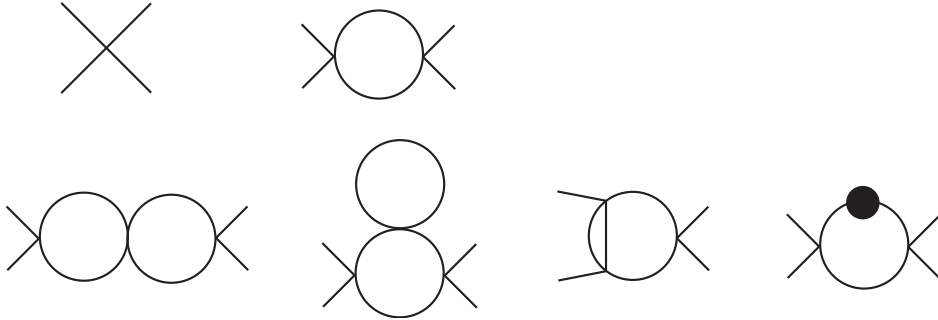


Figure 6: The Feynman diagrams that contribute to h up to order λ^3 . The dot represents an insertion of m_c^2 . The first diagram gives an order λ^1 contribution, the second λ^2 , and the last four λ^3 .

We then find a generalized formula of (2.6) and (2.7), as given by the equations

$$m^{2'} = N^{2\theta-d} e^{-1} [f \Lambda^2 + g (m^2 - m_c^2(\lambda))] , \quad (5.5)$$

$$\lambda' = N^{4\theta-3d} e(\lambda, \frac{m_c^2(\lambda)}{\Lambda^2})^{-2} h(\lambda, \frac{m_c^2(\lambda)}{\Lambda^2}) . \quad (5.6)$$

As in eq. (2.16), we rewrite (5.5) as

$$m^{2'} - m_c^2(\lambda') = N^{2\theta-d} e(\lambda, \frac{m_c^2(\lambda)}{\Lambda^2})^{-1} g(\lambda, \frac{m_c^2(\lambda)}{\Lambda^2}) (m^2 - m_c^2(\lambda)) , \quad (5.7)$$

with

$$m_c^2(\lambda') = N^{2\theta-d} e(\lambda, \frac{m_c^2(\lambda)}{\Lambda^2})^{-1} f(\lambda, \frac{m_c^2(\lambda)}{\Lambda^2}) \Lambda^2 . \quad (5.8)$$

The functional form of $m_c^2(\lambda)$ is determined iteratively in λ by solving the equation (5.8), with (5.6) inserted in the LHS of (5.8). At the order of λ^1 , f is given by the first two diagrams in Figure 4. Then, the solution $m_c^2(\lambda)$ indeed becomes (2.17). At the order of λ^2 , f is given by the last four diagrams in Figure 4. The dot in the loop of the last diagram means the order λ^1 term in $m_c^2(\lambda)$, while the dot in the second diagram means the order λ^2 term in $m_c^2(\lambda)$. e has the order λ^2 term, but it does not affect (5.8) at order λ^2 . The LHS of (5.8) has both contributions from the order λ^1 term in $m_c^2(\lambda')$ with the λ^2 term of (5.6), and from the order λ^2 term in $m_c^2(\lambda')$. In this way, we could obtain the λ^2 term in $m_c^2(\lambda)$. We can also obtain higher order terms iteratively in the same way.

The most important property of eq. (5.8) is that the solution $m_c^2(\lambda)$ is proportional to Λ^2 at all orders in perturbations. Following the same arguments given in section 2, we can see that $m_c^2(\lambda)$ gives the position of the critical line, and (5.6) and (5.7) determine the scaling behavior of the RG flows around the critical line. We therefore find that at all orders in perturbative expansions in λ , the quadratic divergences are completely absorbed in the position of the critical line, and they do not play any role in the dynamics of field theories. The RG flow around the critical line is determined only by the logarithmic divergences. Hence, the assumptions (5.2) and (5.3) we adopted in this section are justified iteratively in λ .

As we mentioned in section 3, eq. (5.7) means that we can separate the subtractive and multiplicative renormalization procedures. Eqs. (5.6) and (5.7) also give an explicit representation of the mass-independent renormalization scheme. The separability of quadratic and logarithmic divergences relies on several properties of loop integrations in Wilsonian RG. First of all, they are free from IR divergences and hence, we can perform the mass expansions. Secondly, the loop integrations are performed from Λ/N to Λ , and thus a divergence

from the subdiagram is compensated by the remaining part of the diagram with negative power of Λ . For instance, in the 5th diagram of Figure 5, while the upper subdiagram gives a Λ^2 contribution, the lower one gives Λ^{-2} , which makes the overall diagram of order Λ^0 and insensitive to the divergence from the subdiagram. We thus need to take care of only the overall superficial degrees of divergences⁶. That is why the naive dimensional analysis is possible.

6 Conclusions and discussions

In this paper, we revisited the hierarchy problem, i.e., stability of mass of a scalar field against large radiative corrections, from the Wilsonian RG point of view. We first saw that quadratic divergences can be absorbed into a position of the critical surface $m_c^2(\lambda)$, and the scaling behavior of RG flows around the critical surface is determined only by logarithmic divergences. The subtraction of the quadratic divergences is unambiguously fixed by the critical surface. In another word, the subtraction is interpreted as taking a new coordinate of the space of parameters such that $m_{\text{new}}^2 = m^2 - m_c^2(\lambda)$. These arguments gave a natural interpretation for the subtractions, and another justification for the subtracted theories as in [3, 4, 5]. The fine-tuning problem, i.e., the hierarchy between the physical scalar mass and the cutoff scale, is then reduced to a problem of taking the bare mass parameter close to the critical surface in taking the continuum limit. It has nothing to do with the quadratic divergences in the theory. Therefore the quadratic divergences are not the real issue of the hierarchy problem. If we are considering a low energy effective theory with an effective cutoff, the subtraction of the quadratic divergences corresponds to taking a boundary condition at the effective cutoff scale. Hence it has nothing to do with the dynamics at a lower energy scale, and when such divergences appear in radiative corrections, we can simply subtract them.

We also considered another type of the hierarchy problem. If a theory consists of multiple physical scales, e.g., the weak scale m_W and the GUT scale m_{GUT} besides the cutoff scale $\tilde{\Lambda}$, a mass of the lower scale m_W receives large radiative corrections $\delta m_W^2 \propto m_{\text{GUT}}^2 \log \tilde{\Lambda}/16\pi^2$ through the logarithmic divergences. Such a mixing of physical mass scales is interpreted as a mixing of

⁶ Our renormalized perturbation is reminiscent of the usual one of BPHZ. In our case, however, the cancellation between the loop contribution and the counter term is not exact. For example, at order λ^1 , the first and the second diagram in Figure 4 did not cancel in (5.8). It follows that, at order λ^2 , the loop contribution of the upper subdiagram in the 4th diagram is not canceled by the dot in the 5th diagram. One can also see the same situation in the last two diagrams in Figure 5. However, divergences from subdiagrams cause no problem in our case.

relevant operators along the RG flows. Unlike the first type of the hierarchy problem, the mass of the larger scale m_{GUT} cannot simply be disposed of by a subtraction. In order to solve such a mixing problem, we need to suppress the mixing by some additional conditions. Of course, if these two scalars are extremely weakly coupled, the mixing can be suppressed. If the couplings are not so weak, we need to cancel the mixing by symmetries or some nontrivial dynamics. A well-known example is the supersymmetry, where the non-renormalization theorem assures the absence of such mixings unless supersymmetry is broken.

Let us comment on scheme dependence of the subtraction. In this paper, we fixed one scheme to perform RG transformations. Then the critical surface is unambiguously determined. If we change the scheme, e.g., from the sharp cut off (2.4) of the higher mode integrations to another one [13], the RG transformations are changed, and so accordingly is the position of the critical surface. But the definition of bare parameters is correlated with the choice of the scheme. Hence a shift of the position of the critical surface by a change of scheme does not mean an ambiguity of the critical surface. Rather it corresponds to changing coordinates of the theory space. In this sense, the subtraction of a position of the critical surface from the bare mass is performed for each fixed scheme without any ambiguity.

What is the meaning of the coefficients of various terms in the bare action? In the investigation of the RG flows, we encountered two kinds of quantities, the mass parameter \tilde{m}^2 and the subtracted mass $(\tilde{m}^2 - \tilde{m}_c(\lambda)^2)$. The issue of the fine-tuning problem, i.e., the stability against the quadratic divergences, is related to which quantity we should consider to be a physical parameter. In the renormalization-group-improved field theory, as we studied in this paper, the subtracted one is considered to be physical. The mass parameter itself depends on a choice of coordinates of the theory space, and changes scheme by scheme. If we want to *derive* low energy field theories from a more fundamental theory, like a string theory, the bare parameter itself, \tilde{m}^2 , is widely believed to be related to the fundamental quantities at the string scale. But since such a quantity is coordinate (of the theory space) dependent, we should use the subtracted one when we relate low energy field theories with more fundamental theories. The amount of subtraction is given by the boundary condition at the cutoff, and determined by the dynamics at higher scales in the fundamental theory. It is independent of the low energy dynamics.

We are thus left with the second type of the hierarchy problem, namely a mixing of the weak scale with another physical scale like m_{GUT} . We classify possible ways out of it:

1. SM up to $\tilde{\Lambda}$
2. New physics around TeV, but nothing beyond up to $\tilde{\Lambda}$
3. New physics at a higher scale, but extremely weakly coupled with SM
4. New physics at a higher scale with nontrivial dynamics or symmetries

The first possibility is to consider a model without any further physical scale up to the cutoff scale $\tilde{\Lambda}$. The Planck scale may play a role of a cutoff scale for the SM. As we saw in this paper, the quadratic divergence of the cutoff order can be simply subtracted and it does not cause any physical effect. In the second possibility, we introduce a new scale which may be coupled with the SM, but suppose that the new scale is not so large compared with the weak scale. Then even if the mixing is not so small, the weak scale does not receive large radiative corrections. Various kinds of TeV scale models are classified into this category. Some examples are ν MSM [9] and the classically conformal⁷ TeV-scale $B-L$ extended model [10]. The third one is to consider a very large physical scale, but with the mutual coupling suppressed to be very small. The final possibilities include a supersymmetric GUT, but the low energy theory of broken supersymmetry must be supplemented with the second type of scenario. If we worry about quadratic divergences, the first three categories need fine-tunings against the cutoff scale and are excluded by the naturalness condition. Hence, most model building beyond the SM has been restricted to the last category. Once we admit that quadratic divergence is not the real issue of the hierarchy problem, it broadens our possibilities of model constructions.

The discussions of the quadratic divergences in this paper can also be extended to the quartic divergences, and if gravity is described in terms of a renormalized field theory, the Wilsonian RG treatment might give a new perspective of the cosmological constant problem. We hope to come back to this issue in future.

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⁷Classical conformality means absence of mass terms for scalar fields. The model of course receives radiative corrections, but classical conformality is not broken by logarithmic divergences.

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